Exercise 2

In Exercises 1–26, solve the following Volterra integral equations by using the *Adomian decomposition method*:

$$u(x) = 6x - x^3 + \int_0^x (x-t)u(t)\,dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\begin{aligned}
\sum_{n=0}^{\infty} u_n(x) &= 6x - x^3 + \int_0^x (x-t)\sum_{n=0}^{\infty} u_n(t)\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= 6x - x^3 + \int_0^x (x-t)[u_0(t) + u_1(t) + \cdots]\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= \underbrace{6x - x^3}_{u_0(x)} + \underbrace{\int_0^x (x-t)u_0(t)\,dt}_{u_1(x)} + \underbrace{\int_0^x (x-t)u_1(t)\,dt}_{u_2(x)} + \cdots
\end{aligned}$$

If we set $u_0(x)$ equal to the function outside the integral, then the rest of the components can be deduced in a recursive manner. After enough terms are written, a pattern can be noticed, allowing us to write a general formula for $u_n(x)$. Note that the $(x-t)$ in the integrand essentially means that we integrate the function next to it twice.

$$\begin{aligned}
u_0(x) &= 6x - x^3 \\
u_1(x) &= \int_0^x (x-t)u_0(t)\,dt = \int_0^x (x-t)(6t - t^3)\,dt = \frac{6x^3}{3\cdot 2} - \frac{x^5}{5\cdot 4} \\
u_2(x) &= \int_0^x (x-t)u_1(t)\,dt = \int_0^x (x-t)\left(t^3 - \frac{t^5}{20}\right)dt = \frac{6x^5}{5\cdot 4\cdot 3\cdot 2} - \frac{x^7}{7\cdot 6\cdot 5\cdot 4} \\
u_3(x) &= \int_0^x (x-t)u_2(t)\,dt = \int_0^x (x-t)\left(\frac{t^5}{20} - \frac{t^7}{840}\right)dt = \frac{6x^7}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} - \frac{x^9}{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4} \\
&\vdots \\
u_n(x) &= \int_0^x (x-t)u_{n-1}(t)\,dt = \frac{6x^{2n+1}}{(2n+1)!} - \frac{6x^{2n+3}}{(2n+3)!}
\end{aligned}$$

The series for $u(x)$ is telescoping because the second term of every component cancels with the first term of the following component. Only two terms remain in the series: $6x$ and the "last" term. Therefore,

$$u(x) = \sum_{n=0}^{\infty}\left[\frac{6x^{2n+1}}{(2n+1)!} - \frac{6x^{2n+3}}{(2n+3)!}\right] = 6x - \underbrace{\lim_{n\to\infty}\frac{6x^{2n+3}}{(2n+3)!}}_{=0} = 6x.$$